

Scalar field cosmology modified by the Generalized Uncertainty Principle

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We consider quintessence scalar field cosmology in which the Lagrangian of the scalar field is modified by the Generalized Uncertainty Principle. We show that the perturbation terms which arise from the deformed algebra are equivalent with the existence of a second scalar field, where the two fields interact in the kinetic part. Moreover, we consider a spatially flat Friedmann-Lemaître-Robertson-Walker spacetime (FLRW), and we derive the gravitational field equations. We show that the modified equation of state parameter w_{GUP} can cross the phantom divide line; that is $w_{GUP} < -1$. Furthermore, we derive the field equations in the dimensionless parameters, the dynamical system which arises is a singular perturbation system in which we study the existence of the fixed points in the slow manifold. Finally, we perform numerical simulations for some well known models and we show that for these models with the specific initial conditions, the parameter w_{GUP} crosses the phantom barrier.

PACS numbers: 98.80.-k, 95.35.+d, 95.36.+x, 11.10.Ef

Keywords: Cosmology; Dark energy; Scalar field; Generalized Uncertainty Principle

1. INTRODUCTION

Various approaches of quantum gravity, such as, String Theory (ST), Doubly Special Relativity (DSR), Black Hole Physics (BHP) predict that there should exist a minimum length scale of the order of the Planck length (l_{PL}), or, in other words, the existence of a maximum energy scale in nature. This prediction motivates to modify the Heisenberg's Uncertainty Principle to some Generalized Uncertainty principle (GUP) in the context of the quantum theory of gravity [1]. The GUP model that has been proposed from String Theory and Black Hole Physics contains the existence of minimum measurable length, whereas DSR provides a GUP model which incorporates the existence of a maximum observable momentum. This maximum momentum scale has been obtained in the context of DSR theory for the consideration of the frame independent invariant upper bound of energy scale into the theory [2]. Interestingly, the above two scenarios, the minimum length scale and the maximum momentum scale can be combined into a single deformed uncertainty principle (DUP), which has the dual nature [3]. However, the modified Heisenberg's algebra that can be generated from this DUP relation is not relativistically covariant. Hence, it is very difficult to apply the DUP in any relativistic scenario. So, here we mainly concentrate on the relativistically covariant modified Heisenberg's algebra, which is constructed from the GUP model that incorporates the minimum length scale only. Indeed, later on, we will see that this modified algebra deforms the coordinate representation of the momentum operator, whereas the position operator remains undeformed.

On the other hand, standard General Relativity (GR) is not enough to describe the phenomena that recently appeared in astrophysics and cosmology, such as, the late-time acceleration of the universe [4–11]. An easy way to explain the accelerated expansion of the universe is to consider an additional fluid which has a negative equation of state parameter (dark energy); this new fluid counteracts the gravitational force, and leads to the observed accelerated expansion. The simplest dark energy candidate is the cosmological constant Λ leading to the Λ CDM cosmology [12–14]. However, this model has two serious problems: the fine tuning problem [14, 15], and the coincidence problem [16]. As a consequence, different dark energy models gave rise into existence, and try to explain the recent accelerating phase of the universe, such as, time-varying $\Lambda(t)$ cosmologies, quintessence, k-essence, phantom and others (for instance see [17–23]).

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Now, it is known from cosmic microwave background observations, that our universe had gone through the inflationary phase at its early stage, and then entered into the radiation, and matter dominated eras, in which the expansion became decelerating. So, at the early stage of high energy regime, it was accelerating, and, in this present low energy regime, it is again accelerating. Quantum gravity effects played an important role at the early universe, see for instance [24–29]. For instance, the mystery behind the origin of magnetic fields with μG (microgauss) strength [30–32] (having approximately the same energy density as the cosmic microwave background radiation) observed in the intergalactic scales, has some explanation in the context of GUP [33]. Furthermore, the existence of a minimum length, specifically GUP, prevent the complete evaporation of black holes [34]. In particular it has been shown that a black hole with temperature greater than the ambient temperature should radiate photons, as well as, ordinary particles until it reaches the Planck size, whereas in the Planck scale, it stops radiating, and, its entropy reaches zero [34] (for a review see [35]).

Thus, it will be very interesting to investigate whether the GUP corrected modified theory can explain the present day accelerating phase, or, not. Because, these theoretical approaches predict that the minimum length effects are quite universal at any stage. Hence, they must have to describe the late time accelerating phase in order to keep the consistency in the theory. Since, it is known that the dark energy, is the proposed candidate to explain the present day acceleration; so, we can study the consequences of the GUP modification on the dark energy models.

We have already mentioned that the modified Heisenberg's algebra corresponding to the GUP model that incorporates the existence of minimum length, modifies the coordinate representation of momentum operator. As a consequence, the dispersion relation of momentum become modified and finally the Klein-Gordon equation deforms accordingly. From this modified Klein-Gordon equation one can easily expect the deformation of a Lagrangian density in field theory. As, in the context of dark energy we have a field theoretic model, namely, the quintessence scalar field models¹ [12, 13, 36–38], so, we can modify it in light of GUP. Therefore, in order to study the late-time acceleration of the universe, in this work, we are interested on the scalar field models (quintessence) in which the Lagrangian has been modified by the GUP. Note that, for this Lagrangian the gravitation equations follows from the Einstein-Hilbert action with the usual variational principle.

From the analysis of the recent data [6–8] it appears that the EoS parameter for the dark energy could have values $w_{DE} \lesssim -1$. The limit $w_{DE} = -1$ is called phantom divide line or phantom barrier. As, the usual quintessence scalar field model can not explain this phase, so, our another motivation is to modify the scalar field model in such a way that it can explain the recent phase. Indeed, here we will show the modified model will drag a correction term into the effective equation of state parameter $w_{GUP}(z)$, and it concludes that, $w_{GUP}(z)$ can cross the above phantom barrier. The plan of the paper is as follows.

In section 2, we give the basic definitions and properties of GUP and we derive the modified Klein-Gordon equation. In section 3, we consider a quintessence scalar field cosmology follows from GUP. We use Lagrange multiplier and we write the gravitational action as that of two scalar field cosmology with a mixed kinetic term. Furthermore, we consider a spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime and write the field equations and the modified EoS. In section 4, we write the field equations in the dimensionless variables, the dynamical system which arises is a slow-fast dynamical system, where we study the fixed points of the system in the slow manifold. Furthermore, we perform numerical simulations for two well known scalar field potentials. Finally, in section 5, we draw our conclusions.

2. GUP-BASED MODIFIED MODEL

The GUP consistent with the existence of the minimum measurable length has the structural form as follows

$$\Delta X_i \Delta P_j \geq \frac{\hbar}{2} [\delta_{ij}(1 + \beta P^2) + 2\beta P_i P_j]. \quad (1)$$

The deformed Heisenberg algebra that can be found from the above GUP model (1) is given by [39–42],

$$[X_i, P_j] = i\hbar [\delta_{ij}(1 + \beta P^2) + 2\beta P_i P_j]. \quad (2)$$

Here β is a parameter of deformation defined by $\beta = \beta_0/M_{Pl}^2 c^2 = \beta_0 \ell_{Pl}^2 / 2\hbar^2$, where M_{Pl} is the Planck mass, ℓ_{Pl} ($\approx 10^{-35}$ m) is the Planck length, $M_{Pl} c^2$ ($\approx 1.2 \times 10^{19}$ GeV) is the Planck energy.

¹ A quintessence scalar field has a time varying equation of state (EoS) parameter $w_\phi(z)$ which is bounded as follows $|w_\phi(z)| \leq 1$, and could describe the acceleration of the universe when $w_\phi(z) < -\frac{1}{3}$.

From the commutation relation (2), the coordinates representation of the momentum operator can be modified to be $P_i = p_i(1 + \beta p^2)$, by keeping $X_i = x_i$ undeformed. Here (x, p) is the canonical representation satisfying $[x_i, p_j] = i\hbar\delta_{ij}$.

In the relativistic four vector form, the above commutation relation (2) can be written as [39, 43]

$$[X_\mu, P_\nu] = -i\hbar[(1 - \beta(\eta^{\alpha\gamma}P_\alpha P_\gamma))\eta_{\mu\nu} - 2\beta P_\mu P_\nu], \quad (3)$$

where the flat metric is given by $\eta_{\mu\nu} \equiv (1, -1, -1, -1)$. The corresponding deformed operators in this case are [43]

$$P_\mu = p_\mu(1 - \beta(\eta^{\alpha\gamma}p_\alpha p_\gamma)), \quad X_\nu = x_\nu, \quad (4)$$

where $p^\mu = i\hbar\frac{\partial}{\partial x_\mu}$, and $[x_\mu, p_\nu] = -i\hbar\eta_{\mu\nu}$.

For a spin-0 particle with rest mass m , the Klein-Gordon equation is $\eta^{\mu\nu}P_\mu P_\nu\Psi - (mc)^2\Psi = 0$. By substituting P_μ from (4), we have the modified Klein Gordon equation has the following form

$$\Delta\Psi + 2\beta\hbar\Delta(\Delta\Psi) + \left(\frac{mc}{\hbar}\right)^2\Psi + O(\beta^2) = 0, \quad (5)$$

where $\Delta = \frac{1}{\sqrt{-g}}\partial_\mu(g^{\mu\nu}\sqrt{-g}\partial_\nu)$ is the Laplace operator, whereas for the metric $g_{\mu\nu} \rightarrow \eta_{\mu\nu}$, $\Delta = \square = \partial_\mu\partial^\mu$. In contrary with the classical Klein-Gordon equation in equation (5), there is the term $2\beta\hbar\Delta(\Delta\Psi)$, which is the quantum correction term. Furthermore, we remark that equation (5) is a fourth-order singular perturbed equation.

Equation (5) could arise from a variational principle, that is, there exists a function $L(\Psi, \Psi_{,\mu}, \Psi_{,\mu\nu}, \dots)$, such that $\mathbf{E}(L) = 0$, where \mathbf{E} is the Euler Lagrange operator. The Lagrangian of the fourth-order Klein-Gordon equation has the following expression [43],

$$L(\Psi, \mathcal{D}_\sigma\Psi) = \frac{1}{2}g^{\mu\nu}\mathcal{D}_\mu\Psi\mathcal{D}_\nu\Psi - \frac{1}{2}\left(\frac{mc}{\hbar}\right)^2\Psi^2, \quad (6)$$

where the new operator \mathcal{D}_μ is $\mathcal{D}_\mu = \nabla_\mu + \varepsilon\nabla_\mu(\Delta)$; ∇_μ is the covariant derivative, i.e. $\nabla_\mu\phi = \phi_{;\mu}$, and $\varepsilon = \beta\hbar^2 = \beta_0\ell_{Pl}^2/2$.

3. GRAVITATIONAL ACTION AND FIELD EQUATIONS

Let a four dimensional Riemannian manifold with metric $g_{\mu\nu}$ and curvature R . We consider the following gravitational action

$$S = S_{GR} + S_{GUP} + S_m, \quad (7)$$

where $S_{GR} = \int d^4x\sqrt{-g}R$ is the Einstein-Hilbert action, S_m describes the matter components, and S_{GUP} is the action of the scalar field modified by the GUP, that is,

$$S_{GUP} = - \int \sqrt{-g} \left(\frac{1}{2}g^{\mu\nu}\mathcal{D}_\mu\phi\mathcal{D}_\nu\phi - V(\phi) \right). \quad (8)$$

Therefore, the gravitational action is²:

$$S = \int \sqrt{-g} d^4x \left(R - \frac{1}{2}g^{\mu\nu}\mathcal{D}_\mu\phi\mathcal{D}_\nu\phi + V(\phi) \right) + S_m,$$

or, equivalently,

$$S = S_{GR} + S_\phi + S_m + \int \sqrt{-g} d^4x \left(-\varepsilon g^{\mu\nu}\phi_{;(\mu}g^{ab}\phi_{;|ab|\nu)} \right), \quad (9)$$

² In contrary to the Saez-Ballester scalar field model [44], in which the operator \mathcal{D}_μ , can be seen as $f(\phi)\nabla_\mu$, where ∇_μ , is the covariant derivative, the new operator \mathcal{D}_μ includes second-order derivatives. For the cosmological viability of the Saez-Ballester model see [45].

where S_ϕ is the action of the “classical” scalar field [23]. Note that, the last term of the above action (9) can be written equivalently as follows

$$S_C = \int \sqrt{-g} d^4x (\varepsilon g^{\mu\nu} \phi_{;(\mu\nu)} g^{ab} \phi_{;ab}). \quad (10)$$

The action integral (9) provide us with a higher-order theory of gravity in which the higher-order terms follow from the term S_C .

In order to simplify the calculations, we introduce a Lagrange multiplier³, and, consider the new variable ψ defined by $\psi = g^{\mu\nu} \phi_{; \mu\nu}$; therefore, incorporating the Lagrange multiplier λ into (10), and by solving the Lagrange’s equation $\frac{\partial S}{\partial \psi} = 0$, we have $\lambda = -2\varepsilon\psi$. Hence, applying integration by parts, equation (10) becomes

$$S_C = \int \sqrt{-g} d^4x (-2\varepsilon g^{ab} \phi_{;a} \psi_{;b} - \varepsilon \psi^2). \quad (11)$$

Therefore, the gravitational action (9) becomes

$$S = S_{GR} + S_\phi + S_\psi + S_m, \quad (12)$$

in which,

$$S_\psi = \int \sqrt{-g} d^4x [-2\varepsilon g^{ab} \phi_{;a} \psi_{;b} - \varepsilon \psi^2]$$

The last action describes a field which is coupled with the field ϕ , where the extra term follows from the perturbation effects of GUP.

Finally, from action (12), we have that the gravitation equations are

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}(\phi, \psi) + 8\pi G \bar{T}_{\mu\nu}, \quad (13)$$

where $T_{\mu\nu}(\phi, \psi)$ is the energy-momentum tensor for the two fields⁴ $\{\phi, \psi\}$, and, $\bar{T}_{\mu\nu}$ is the energy- momentum of the matter component. We remark that the kinetic metric which is defined by the two fields $\{\phi, \psi\}$ can not be written in a diagonal form, since the “coordinate” transformation $\{\phi, \psi\} \rightarrow \{\bar{\phi}, \bar{\psi}\}$ which diagonalizes the kinetic metric depends on ε^{-1} , i.e. it is singular transformation. Which means that the action (12) can not be written in that of the Quintom model [46, 47]. Furthermore, the second fluid ψ , is not a real fluid but it follows from the higher-order derivatives of the field equations.

In particular, the reduction of the order of the field equations from a fourth-order theory to a second-order, by using a Lagrange multiplier, it means that in the same time the number of the dependent variables is increased. An other analogue is the $f(R)$ -gravity in the metric formalism [48]. The last gravitational theory for nonlinear $f(R)$ function is reduced with the use of the Lagrange multiplier in a Brans-Dicke theory with vanished Brans-Dicke parameter [49], the so called O’Hanlon massive dilaton theory [50]. For other applications of Lagrange multipliers in different theories of gravity, see [51–53], and for a discussion on the degrees of freedom in alternative theories of gravity, see [54].

In the following section we continue with the derivation of the field equations in a spatially flat FLRW spacetime.

3.1. Field equations in a FLRW background

Let us consider a spatially flat FLRW spacetime with line element

$$ds^2 = dt^2 - a^2(t) (dx^2 + dy^2 + dz^2), \quad (14)$$

and, with gravitational action (12), where the matter tensor $\bar{T}_{\mu\nu}$ describes a dust fluid with energy density ρ_m . Moreover, from the conservation law $\bar{T}^{\mu\nu}_{;\nu} = 0$ (Bianchi identity), and, for the space (14), we have that $\rho_m = \rho_{m0} a^{-3}$,

³ Lagrange multipliers are useful to reduce to order of the differential equations. However at the same time, the dimension of the dependent variables is increased. For applications of the Lagrange multipliers in high order theories of gravity, see for instance [51–53]

⁴ For exact solutions of two scalar field models, see [56, 57], and for a quintom scenario with mixed kinetic term, see [58].

where ρ_{m0} is the present value of the energy density. From the cosmological principle, we have that the fields $\{\phi, \psi\}$ inherit the symmetries of the metric (14), hence, $\phi = \phi(t)$, $\psi = \psi(t)$.

Therefore, for the space (14), the gravitation equations (13) follow from the following Lagrangian⁵

$$L = 3a\dot{a}^2 - \frac{1}{2}a^3\dot{\phi}^2 - 2\varepsilon a^3\dot{\phi}\dot{\psi} + a^3V(\phi) - \varepsilon a^3\psi^2 + \rho_{m0}. \quad (15)$$

Lagrangian (15) can be seen as the Lagrangian of a particle moving in a Riemannian manifold with metric γ_{ij} , and effective potential $V_{eff} = - (a^3V(\phi) - \varepsilon a^3\psi^2 + \rho_{m0})$, that is,

$$L(y^j, \dot{y}^j) = \frac{1}{2}\gamma_{ij}\dot{y}^i\dot{y}^j - V_{eff}(y^k), \quad (16)$$

where, $y^i = (a, \phi, \psi)$, and γ_{ij} is given by the following line element

$$ds_\gamma^2 = 6ada^2 - a^3d\phi^2 - 4\varepsilon a^3d\phi d\psi. \quad (17)$$

For the Euler-Lagrange equations of (16) we have that these are given by the following expression

$$\gamma_{ik}\ddot{y}^k + \gamma_{ir}\Gamma_{jk}^r\dot{y}^j\dot{y}^k + (V_{eff})_{,i} = 0, \quad (18)$$

where Γ_{jr}^k are the Christoffel symbols of γ_{ij} , and the nonzero Christoffel symbols are

$$\Gamma_{aa}^a = \frac{1}{2a}, \quad \Gamma_{\phi\phi}^a = \frac{a}{4}, \quad \Gamma_{\phi\psi}^a = \frac{\varepsilon}{2}a, \quad (19)$$

and

$$\Gamma_{a\phi}^\phi = \frac{3}{2a}, \quad \Gamma_{a\psi}^\psi = \frac{3}{2a}. \quad (20)$$

Therefore from (18)-(20), we have that the second Friedmann's equation which is the Euler-Lagrange equation of (15) with respect to the scale factor, a , is given as follows

$$\ddot{a} + \frac{1}{2a}\dot{a}^2 + \frac{a}{4}\dot{\phi}^2 + a\varepsilon\dot{\phi}\dot{\psi} - \frac{1}{2}a(V(\phi) - \varepsilon\psi^2) = 0, \quad (21)$$

and the Klein-Gordon equations for the “two field” $\{\phi, \psi\}$, are

$$\ddot{\phi} + \frac{3}{a}\dot{a}\dot{\phi} - \psi = 0, \quad (22)$$

$$\varepsilon\left(\ddot{\psi} + \frac{3}{a}\dot{a}\dot{\psi}\right) + \frac{1}{2}(\psi + V_{,\phi}) = 0, \quad (23)$$

which follow from the conservation law $T^{\mu\nu}_{;\nu}(\phi, \psi) = 0$. Furthermore, the first Friedmann's equation can be seen as the Hamiltonian function which follows from Lagrangian (15), and it is⁶,

$$\rho_{m0} = 3a\dot{a}^2 - \frac{1}{2}a^3\dot{\phi}^2 - 2\varepsilon a^3\dot{\phi}\dot{\psi} - a^3V(\phi) + a^3\varepsilon\psi^2. \quad (24)$$

Here, we would like to note that, the set of the field equations (24)-(23) form a singular perturbation dynamical system. Moreover, it is straightforward to prove that, equation (22) is the constrained equation $\psi = g^{\mu\nu}\phi_{;\mu\nu}$, and, if we replace ψ from (22) in (23), then we will have a fourth-order equation, which is the modified Klein-Gordon equation for the field, ϕ , in GUP. Furthermore, in the limit in which $\varepsilon = 0$, that is, the deformation parameter β is vanished and we have the classical uncertainty principle, Lagrangian (15) is that of quintessence scalar field with dust fluid, and in the same time equations (21)-(23) reduced to the field equations for the last model.

⁵ In the following, we consider $8\pi G = 1$.

⁶ Alternatively equation (24) can be derived from the Euler-Lagrange equation with respect to a new variable, N , which arises from the lapsed time $dt = Nd\tau$, for instance see [55].

An alternative way to write the field equations is,

$$3H^2 = \rho_{GUP} + \rho_m, \quad (25)$$

$$2\dot{H} + 3H^2 = -p_{GUP}, \quad (26)$$

where $\rho_{GUP} = \rho_\phi + \rho_C$, $p_{GUP} = p_\phi + p_C$. Here, ρ_ϕ , p_ϕ are the density and pressure of the usual scalar field given by

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi), \quad p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi), \quad (27)$$

and ρ_C , p_C are the perturbation terms

$$\rho_C = \varepsilon \left(2\dot{\phi}\dot{\psi} - \psi^2 \right), \quad p_C = \varepsilon \left(2\dot{\phi}\dot{\psi} + \psi^2 \right). \quad (28)$$

However, the two fluids $\{\rho_\phi, \rho_C\}$ are interacting; furthermore, the field ρ_C , i.e. equation (28) is not a real field, but it was introduced by the quantum corrections of GUP.

Finally, the effective EoS parameter for the scalar field which follows from GUP is

$$w_{GUP} = \frac{\left(\frac{1}{2}\dot{\phi}^2 - V(\phi) \right) + \varepsilon \left(2\dot{\phi}\dot{\psi} + \psi^2 \right)}{\left(\frac{1}{2}\dot{\phi}^2 + V(\phi) \right) + \varepsilon \left(2\dot{\phi}\dot{\psi} - \psi^2 \right)}. \quad (29)$$

Using the Taylor series expansion of equation (29) at $\varepsilon \rightarrow 0$, we have $w_{GUP} = w_\phi + w'$, where $w_\phi = \frac{p_\phi}{\rho_\phi}$, is the EoS parameter for the “classical” scalar field and

$$w' = \frac{1}{\rho_\phi} (p_C - w_\phi \rho_C) + O(\varepsilon^2). \quad (30)$$

In the limit where $\dot{\phi}^2 \ll V$, which means that $w_\phi \simeq -1$, then $w' = \varepsilon \frac{4}{V} \dot{\phi}\dot{\psi}$; therefore w_{GUP} becomes

$$w_{GUP} \simeq -1 + \frac{4\varepsilon}{V} \dot{\phi}\dot{\psi} \lesssim -1, \quad \text{when } \dot{\phi}\dot{\psi} \leq 0. \quad (31)$$

Hence, the EoS parameter w_{GUP} can cross the phantom barrier provided when $\dot{\phi}\dot{\psi} < 0$, and, $|\dot{\psi}| \gg 1$. Here, we would like to note that, since the field equations form a singular perturbation system (slow-fast system), it is possible that $|\dot{\psi}| \gg 1$, which means that either when the parameter ε is very small, the product $|\varepsilon\dot{\psi}|$ could be big enough in order the perturbation effects to be measurable. In order to show this, in the following section we study the existence of fixed points for the singular perturbation system in the slow manifold. Finally, we perform numerical simulations for some well known scalar field models, and, we show that, w_{GUP} can cross the phantom divide line.

4. COSMOLOGICAL EVOLUTION

In this section, we consider the field equations (13) where now the energy momentum tensor $\bar{T}_{\mu\nu}$ is $\bar{T}_{\mu\nu} = \bar{T}_{\mu\nu}^{(m)} + \bar{T}_{\mu\nu}^{(r)}$ where $\bar{T}_{\mu\nu}^{(m)}$, $\bar{T}_{\mu\nu}^{(r)}$, are the energy-momentum tensors for a dust fluid ($w_m = 0$) (dark matter), and a radiation fluid ($w_r = \frac{1}{3}$) respectively.

The field equations (13) for the spatially flat FLRW spacetime are

$$3H^2 = \rho_\phi + \rho_m + \rho_r + \varepsilon \left(2\dot{\phi}\dot{\psi} - F(\psi) \right), \quad (32)$$

$$2\dot{H} + 3H^2 = -p_\phi - p_r - \varepsilon \left(2\dot{\phi}\dot{\psi} + F(\psi) \right), \quad (33)$$

where $F(\psi) = \psi^2$.

We define the dimensionless variables [59, 60]

$$x_1 = \frac{\dot{\phi}}{\sqrt{6}H}, \quad x_2 = \frac{\sqrt{V(\phi)}}{\sqrt{3}H}, \quad x_3 = \frac{\sqrt{\rho_r}}{\sqrt{3}H}, \quad (34)$$

$$y_1 = \frac{2\sqrt{6}}{3} \frac{\dot{\psi}}{H}, \quad y_2 = \frac{\sqrt{F}}{\sqrt{3}H}, \quad \Omega_m = \frac{\rho_m}{3H^2}. \quad (35)$$

In the new variables, the first Friedmann's equation (32) can be written as follows

$$\Omega_m = 1 - \Omega_{GUP} - \Omega_r, \quad (36)$$

where now $\Omega_r = (x_3)^2$, and $\Omega_{GUP} = \left[(x_1)^2 + (x_2)^2 \right] + \varepsilon \left[x_1 y_1 - (y_2)^2 \right]$.

Furthermore, the second Friedmann's equation (33) takes the form $\frac{\dot{H}}{H^2} = \frac{3}{2}\Xi(\mathbf{x}, \mathbf{y}, \varepsilon)$, where

$$\Xi(\mathbf{x}, \mathbf{y}, \varepsilon) \equiv -1 - \left[(x_1)^2 - (x_2)^2 \right] - \varepsilon \left[x_1 y_1 + (y_2)^2 \right] - \frac{1}{3} (x_3)^2,$$

hence, the total EoS parameter is $w_{eff} = -1 - \Xi(\mathbf{x}, \mathbf{y}, \varepsilon)$.

The EoS parameter (29) for the scalar field in the new variables becomes

$$w_{GUP} = \frac{\left[(x_1)^2 - (x_2)^2 \right] + \varepsilon \left(x_1 y_1 + (y_2)^2 \right)}{\left[(x_1)^2 + (x_2)^2 \right] + \varepsilon \left(x_1 y_1 - (y_2)^2 \right)}, \quad (37)$$

where in the limit $w_\phi \rightarrow -1$, becomes

$$w_{GUP} = -1 + 2\varepsilon \frac{x_1 y_1}{(x_2)^2} + O(\varepsilon^2). \quad (38)$$

The derivation of the variables \mathbf{x}, \mathbf{y} with respect to the lapse time $N = \ln a$ gives the following dynamical system

$$\frac{d\mathbf{x}}{dN} = \mathbf{f}(\mathbf{x}, \mathbf{y}, \lambda, \mu, \varepsilon), \quad \frac{d\lambda}{dN} = h_1(\mathbf{x}, \mathbf{y}, \lambda, \mu, \varepsilon), \quad (39)$$

$$\frac{dy_2}{dN} = g_2(\mathbf{x}, \mathbf{y}, \lambda, \mu, \varepsilon), \quad \frac{d\mu}{dN} = h_2(\mathbf{x}, \mathbf{y}, \lambda, \mu, \varepsilon), \quad (40)$$

$$\varepsilon \frac{dy_1}{dN} = g_1(\mathbf{x}, \mathbf{y}, \lambda, \mu, \varepsilon), \quad (41)$$

where the new variables λ, μ are $\lambda = \left(-\frac{V_{,\phi}}{V} \right)$, $\mu = \left(-\frac{1}{2} \frac{F_{,\psi}}{F} \right)$, and, the functions \mathbf{f}, \mathbf{g} and \mathbf{h} are as follows:

$$f_1 \equiv -3x_1 - \frac{\sqrt{6}}{2} \mu (y_2)^2 - \frac{3}{2} x_1 \Xi. \quad (42)$$

$$f_2 \equiv -\frac{\sqrt{6}}{2} \lambda x_1 x_2 - \frac{3}{2} x_2 \Xi. \quad (43)$$

$$f_3 \equiv -2x_3 - \frac{3}{2} x_3 \Xi. \quad (44)$$

$$g_1 \equiv -3\varepsilon y_1 + \sqrt{6} \lambda (x_2)^2 + \sqrt{6} \mu (y_2)^2 - \frac{3}{2} \varepsilon y_1 \Xi. \quad (45)$$

$$g_2 \equiv -\frac{\sqrt{6}}{4} \mu y_1 y_2 - \frac{3}{2} y_2 \Xi. \quad (46)$$

$$h_1 \equiv -\sqrt{6} \lambda^2 (\Gamma_1 - 1) x_1; \quad \Gamma_1 = \frac{V_{,\phi\phi} V}{V_{,\phi}^2}. \quad (47)$$

$$h_2 \equiv -\frac{\sqrt{6}}{2} \mu^2 (\Gamma_2 - 1) y_1; \quad \Gamma_2 = \frac{F_{,\psi\psi} F}{F_{,\psi}^2}. \quad (48)$$

4.1. Slow-fast dynamical system

The dynamical system (39)-(41) is a singular perturbation system; specifically, it is a slow-fast system of first-order ordinary differential equations, where y_1 is the fast variable [61–64]. Under the transformation $\bar{N} = \varepsilon N$, the dynamical system (39)-(41) becomes

$$\frac{d\mathbf{x}}{d\bar{N}} = \varepsilon \mathbf{f}, \quad \frac{d\lambda}{d\bar{N}} = \varepsilon h_1, \quad \frac{dy_1}{d\bar{N}} = g_1, \quad \frac{dy_2}{d\bar{N}} = \varepsilon g_2, \quad \frac{d\mu}{d\bar{N}} = \varepsilon h_2, \quad (49)$$

TABLE I: Fixed points analysis of the evolution equations (39)-(41) in the slow manifold for the exponential potential

Point	$(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$	$(\mathbf{y}_1, \mathbf{y}_2)$	μ	Eigenvalues	Stability	Ω_m	Ω_r	Ω_{GUP}	w_{eff}	w_{GUP}
O	$(0, 0, 0)$	$(0, 0)$	μ	$\frac{3}{2}(-1, 1, 1, 0)$	Unstable	1	0	0	0	\nexists
O^*	$(0, 0, 0)$	$(y_1, 0)$	0	$\frac{3}{2}(-1, 1, 1, 0)$	Unstable	1	0	0	0	\nexists
A_{\pm}	$(\pm 1, 0, 0)$	$(0, 0)$	μ	$\left(3, 3, 3 \mp \frac{3\lambda}{\sqrt{6}}, 0\right)$	Unstable	0	0	1	1	1
A_{\pm}^*	$(\pm 1, 0, 0)$	$(y_1, 0)$	0	$\left(3, 3, 3 \mp \frac{3\lambda}{\sqrt{6}}, 0\right)$	Unstable	0	0	1	1	1
R_{\pm}	$(0, 0, \pm 1)$	$(0, 0)$	μ	$(-1, 2, 2, 0)$	Unstable	0	1	0	$\frac{1}{3}$	\nexists
R_{\pm}^*	$(0, 0, \pm 1)$	$(y_1, 0)$	0	$(-1, 2, 2, 0)$	Unstable	0	1	0	$\frac{1}{3}$	\nexists

which is a regular perturbation system [62]. The two dynamical systems (39)-(41) and (49) are equivalent but in different “time scales”.

In order to continue, we will use the Tikhonov’s theorem [61, 64], and, we will study the fixed points of the dynamical system (39)-(41) in the slow manifold $g_2(\mathbf{x}, \mathbf{y}, \lambda, \mu, 0) = 0$. This solution is called outer solution of the singular perturbed system.

In the limit $\varepsilon = 0$, the dynamical system is equivalent to the algebraic dynamical system (39), (40) and $g_2(\mathbf{x}, \mathbf{y}, \lambda, \mu, 0) = 0$; the last gives the slow manifold $\lambda(x_2)^2 + \mu(y_2)^2 = 0$. Furthermore, when $\varepsilon = 0$, from (36), we have that $\mathbf{x}\mathbf{x}^T \leq 1$; that is, for the variables \mathbf{x} , $\mathbf{x} \in [-1, 1]$ holds. Moreover, since $V(\phi) \geq 0$, the parameter x_2 is constrained as follows: $x_2 \in [0, 1]$.

We consider the exponential potential $V(\phi) = V_0 e^{-\sigma\phi}$, where σ is a constant different from zero, then $\lambda = \sigma$; hence we have that $\Gamma_1 = 1$, $\Gamma_2 = \frac{1}{2}$. The fixed points of the outer solution with the physical value of the problem and the corresponding eigenvalues are given in table I.

The fixed points of the table I exist for all values of the parameter λ . Points O, O^* correspond to the matter epoch, where the total EoS parameter is $w_{tot} = 0$. At the points A_{\pm}, A_{\pm}^* , the universe is dominated by the kinetic energy of the scalar field ($\Omega_{\phi} = 1$, $V(\phi) = 0$), which means that the scalar field acts as a stiff fluid, i.e. $\rho_{\phi} = \rho_{\phi 0} a^{-6}$, this solution corresponds to a non-accelerated universe. The fixed points R_{\pm}, R_{\pm}^* correspond to the radiation epoch, where $\Omega_r = 1$ and $w_{tot} = \frac{1}{3}$. Finally, the linearized system (39), (40) at the fixed points admits positive eigenvalues; that is, all the fixed points are unstable.

However, in contrary to the “classical” minimal coupled scalar field cosmology, there are not the scalar field dominated and the tracker solutions [23]. We would like to note that, this analysis holds for very small values of the parameter $\Delta N = N - N_0$; where N_0 is the value which we consider as a starting point.

When $\varepsilon = 0$, the regular perturbation system (49) gives that the parameters $\mathbf{x}, \lambda, y_2, \mu$ are constants (that holds for any potential), and, the only non-constant equation is

$$\frac{dy_1}{dN} = \sqrt{6}\lambda(x_2)^2 + \sqrt{6}\mu(y_2)^2 = \alpha = \text{const}, \quad (50)$$

that is, $y_1(\bar{N}) = \alpha\bar{N} + \beta$, where $\alpha, \beta \in \mathbb{R}$. However, since $\varepsilon = 0$, the cosmological parameters are independent on y_1 ; that is, for very large values of the parameter $(N - N_0) = \ln a$, the cosmological parameters Ω_m, Ω_r , etc. will be constants.

In order to understand better the effects of the GUP in the dark energy models, we perform numerical simulations for the dynamical system (39)-(41) for the power law model $V(\phi) = V_0 \phi^{-1}$ [23], and, the hyperbolic model $V(\phi) = V_0 (\cosh(\sigma\phi) - 1)^p$ [36].

In fig. 1, we give the numerical solutions of the energy density evolution, and of the EoS parameter w_{GUP} for the potential $V(\phi) = V_0 \phi^{-1}$. For the non-modified scalar field, the same plots can be found in [23]. We observe that in contrary with the “classical” quintessence scalar field, where in the tracking solution, the EoS parameter has the value $w_{\phi} = -\frac{2}{3}$, whereas the w_{GUP} can cross the phantom divide line; however, the energy density evolution of the two models has similar behavior.

Furthermore, in fig. 2, we compare the evolution of the EoS parameters w_{ϕ} and for the modified EoS parameter w_{GUP} for the hyperbolic model $V(\phi) = V_0 (\cosh(\sigma\phi) - 1)^p$. We observe that the scalar field modified by GUP can cross the phantom divide line, i.e. $w_{GUP} < -1$.

Finally, in order to understand the behavior of the slow-fast system (39)-(41), in fig. 3, we give the evolution of the parameter $\left|\varepsilon \frac{dy_1}{dN}\right|$ for the hyperbolic potential. From the fig. 3, we observe that for large ΔN parameter, y_1 is fast, i.e. $y_1 \simeq \varepsilon^{-1}$, which means that the term εy_1 in equation (38) can be measurable. For the numerical solutions, in figs. 1, 2 and 3, we selected $\varepsilon = 10^{-70}$, which corresponds to the natural value of the parameter β_0 , i.e. $\beta_0 \simeq 1$. However,

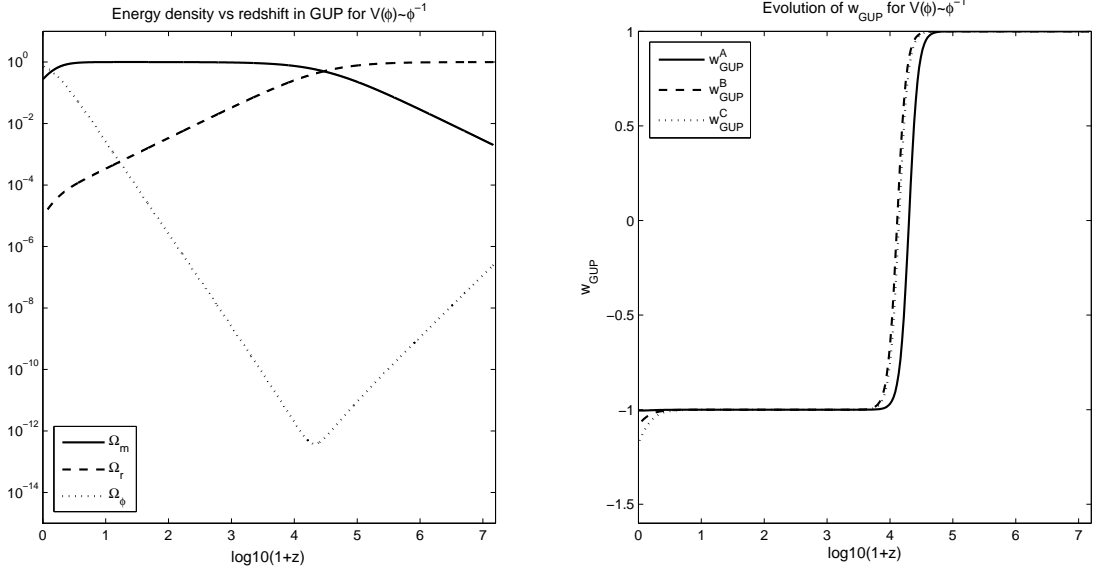


FIG. 1: Numerical solution of the field equation in the dimensionless variables for minimally coupled scalar field cosmology modified by the GUP for the power law potential $V(\phi) = V_0\phi^{-1}$. The left fig. is the evolution of the energy density of the Ω_m (solid line), Ω_r (dashed line) and Ω_ϕ (dot line). For the numerical simulation we start from points in the radiation epoch, $\log_{10}(1+z) = 7.21$. The right fig. is the evolution of the scalar field equation of state parameter (EoS) for the same model. For the numerical integration, we have considered $\beta_0 = 1$.

we would like to remark that the behavior of the solutions is similar and for smaller values of the parameter ε , i.e. β_0 ; since when the parameter ε decrease, the fast variable y_1 becomes “faster”.

5. CONCLUSIONS

In this work, we have considered a modified quintessence scalar field Lagrangian which follows from the GUP. The modified Klein-Gordon equation for the scalar field admits a fourth-order perturbation term which follows from the quantum corrections. Furthermore, we have considered the gravitational action, and, with the use of Lagrange multiplier, we rewrote the gravitational action as that of a two scalar fields model with a mixed kinetic term; we remark that, the second field was introduced by the quantum corrections of GUP and it is not a real field. The proposed model, i.e. the action integral (12), it can be compared with that of a quintom model in which two fields take place. However, in our consideration there exist only one field which satisfies a fourth-order Klein-Gordon equation and in general it is different from the quintom models, moreover the coordinate transformation which diagonalize the kinetic term, which is defined by the field ϕ , and the Lagrange multiplier ψ , is a singular transformation; which means that the inverse transformation can not be defined.

Consider now the action integral of a quintom model with mixed kinetic term, which has been introduced in [58],

$$S_{mix} = \int d^4x \left[R - \frac{\phi_0}{2} g^{\mu\nu} \phi_{;\mu} \phi_{;\nu} + \frac{\psi_0}{2} g^{\mu\nu} \psi_{;\mu} \psi_{;\nu} - \frac{\alpha}{2} g^{\mu\nu} \phi_{;\mu} \psi_{;\nu} + V_1(\phi) + V_2(\psi) \right] \quad (51)$$

where ϕ_0, ψ_0, α are constants⁷, then in comparison with the action integral (12), it can be seen that, $\psi_0 = 0$, $V_2(\psi)$ is fixed, which follows from the Lagrange multiplier, and the constant, α , is related with the deformation parameter of the GUP, i.e. $\alpha \simeq \beta$. Hence a quintom scenario with mixed terms in which the parameter, α , is small, can arise from the GUP.

In the case of a spatially flat FLRW spacetime, we derived the field equations and we wrote the modified cosmological parameters. We show that the EoS parameter w_{GUP} for the scalar field has not a minimum boundary; that is, it

⁷ In [58], $\phi_0 = \psi_0 = 1$

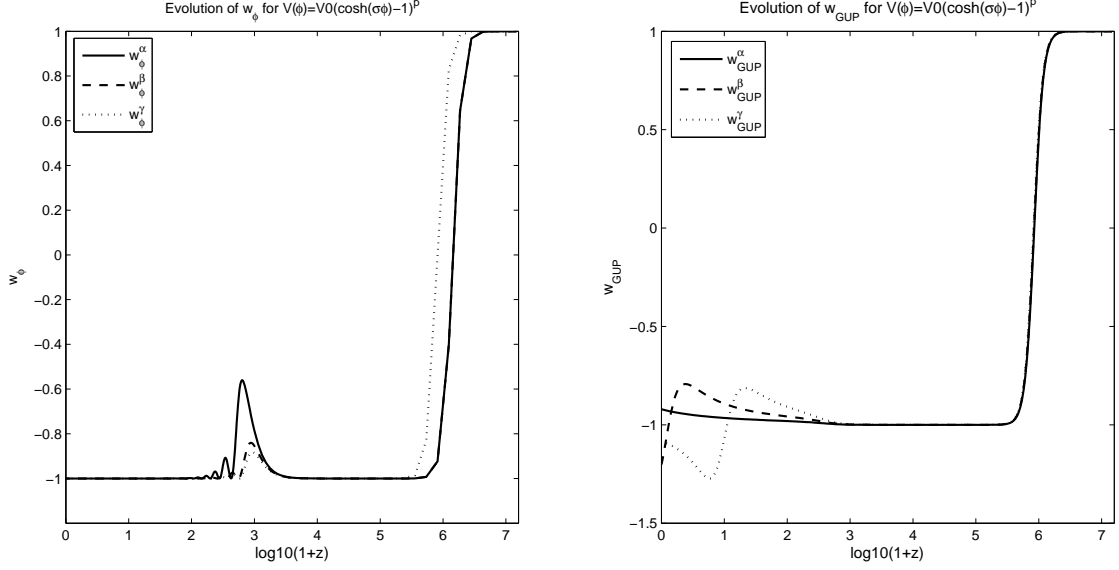


FIG. 2: Numerical solution of the EoS parameter w_ϕ and w_{GUP} for minimally coupled scalar field cosmology for the hyperbolic potential $V(\phi) = V_0 (\cosh(\sigma\phi) - 1)^p$ with $p = 0.2$, and, $\sigma = 20$ [36]. The left fig. describes the evolution of the EoS parameter w_ϕ for the “classical” scalar field, whereas the right fig. is for the modified EoS parameter w_{GUP} . For the numerical simulation, we start from points in the radiation epoch, $\log_{10}(1+z) = 7.21$, and, we consider $\beta_0 = 1$.

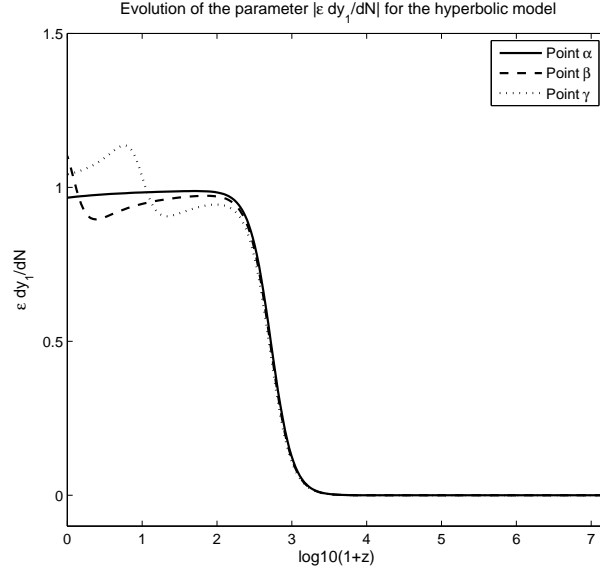


FIG. 3: Numerical solution of the parameter $\epsilon dy_1/dN$ (equation (41)) for the hyperbolic model of fig.2. We observe that, for large values of ΔN , the fast variable y_1 take values $y_1 \simeq \epsilon^{-1}$, and, when $\epsilon dy_1/dN > 1$, the w_{GUP} crosses the phantom divide line, which means that the term ϵy_1 of equation (38) is measurable. For the numerical integration, we have considered $\beta_0 = 1$. However, a similar behavior holds, and for different values of β_0 .

is possible to cross the phantom divide line, such as, $w_{GUP} < -1$. Moreover, in order to study the cosmological evolution of the model, we wrote the field equations in the dimensionless variables, and, in the lapse time $N = \ln a$. Since the dynamical system is a singular perturbation system we studied the existence of fixed points in the slow manifold. We have shown that, for an exponential potential, the outer solution of the field equations admits only unstable fixed points, which correspond to the matter dominated era, the radiation epoch, and, to the solution where the universe is dominated by the kinetic part of the scalar field. For “large” values of the lapse time, we demonstrate

that the cosmological parameters are constants.

Finally, we have performed numerical simulations of the field equations for two well known potentials $V(\phi)$; in both cases, the EoS parameter w_{GUP} can take values lower than minus one. That result is important, since we can have a phantom behavior which follows for the quantum corrections of a quintessence scalar field model.

Acknowledgments

The authors would like to thank the anonymous referees for various comments and suggestions which improved the quality of the present work. AP acknowledges financial support of INFN. S. Pan acknowledges CSIR, Govt. of India for research grants through SRF scheme (File No: 09/096(0749)/2012-EMR-I).

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